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Inequalities in Entanglement Percolation

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We present proofs of two results concerning entanglement in three-dimensional bond percolation. Firstly, the critical probability for entanglement with free boundary conditions is strictly less than the critical probability for connectivity percolation. (The proof presented here is a detailed justification of the ideas sketched in Aizenman and Grimmett.) Secondly, under the hypothesis that the critical probabilities for entanglement with free and wired boundary conditions are different, for p between the two critical probabilities, the size of the entangled cluster at the origin with free boundary conditions does not have exponentially decaying tails.

KEY WORDS: Percolation; entanglement; critical probability; boundary condition.

1. INTRODUCTION

In the bond percolation model, edges of the three-dimensional cubic lattice are declared independently *open* with probability p, and *closed* with probability $1-p$. Roughly speaking, a graph in three-dimensional space is said to be entangled if it cannot be "pulled apart" when its edges are regarded as physical connections made of elastic. Entanglement percolation is concerned with the study of open entangled graphs in the percolation model.

Entanglement in percolation appears to have been first studied in ref. 2, using partly non-rigorous methods. In that paper, an "entanglement critical probability" p_e for the existence of infinite open entangled graphs is investigated; the numerical estimate $p_c - p_e \approx 1.8 \times 10^{-7}$ is derived, although no formal definition is given for an entangled graph. (Here p_c is the usual percolation critical probability for the existence of infinite open connected

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graphs). In ref. 1 the authors describe how their general method can be used to obtain the strict inequality $p_e < p_c$, but again no definition of p_e is given. A rigorous theory of entanglement was developed later in ref. 3. It turns out that significant topological issues are involved, and in particular there are several non-equivalent natural definitions of entanglement for infinite graphs. Two such definitions correspond in a natural way with "free" and "wired" boundary conditions respectively, and these give rise to two potentially different critical probabilities p_e^0 and p_e^1.

The following inequalities hold:

$$0 < p_e^1 \leqslant p_e^0 < p_c < 1.$$

The inequality $0 < p_e^1$ is proved in ref. 4. The inequality $p_e^0 \leqslant p_c$ is a triviality. In this article we shall present a proof of the strict inequality $p_e^0 < p_c$, based on the sketch given in ref. 1. The inequality $p_e^1 \leqslant p_e^0$ is a triviality. It is a very interesting unsolved problem to decide whether $p_e^1 = p_e^0$. (The analogous question for rigidity was answered affirmatively in ref. 5.) We shall prove the following result, which makes it plausible that the above equality holds. Suppose on the contrary that $p_e^1 < p_e^0$, and let $p_e^1 < p < p_e^0$. Then under the definition of entanglement corresponding to p_e^0, the size of the entangled cluster at the origin has tails which decay more slowly than exponentially. This would be at odds with the natural conjecture that for all $p < p_e^0$, such tails decay exponentially (as in connectivity percolation).

Additional material on entanglement percolation appears in refs. 6–7.

2. NOTATION AND RESULTS

We start with some definitions. The three-dimensional cubic lattice is the graph with vertex set $\mathbb{Z}^3$ and edge set

$$\mathbb{L} = \{\{x, y\} \subseteq \mathbb{Z}^3 : \|x - y\| = 1\}$$

where $\|\cdot - \cdot\|$ denotes Euclidean distance. The *origin* is the vertex $O = (0, 0, 0) \in \mathbb{Z}^3$. In the bond percolation model with parameter p, each edge in $\mathbb{L}$ is declared *open* with probability p, and *closed* otherwise, independently for different edges. More formally, we consider the product probability measure P_p with parameter p on the probability space $\{0, 1\}^{\mathbb{L}}$, and we write E_p for the corresponding expectation operator. An element ω of the probability space is called a *configuration*, and an edge $e \in \mathbb{L}$ is said to be *open* if $\omega(e) = 1$ and *closed* if $\omega(e) = 0$. It is convenient to define a *graph* to be a subset of $\mathbb{L}$. We write $W = W(\omega)$ for the random graph of all open edges.

Percolation theory is concerned with the existence of infinite connected components. We define

$$\theta(p) = P_p(W \text{ has an infinite connected component containing } O)$$

and

$$p_c = \sup\{p\colon \theta(p) = 0\}.$$

For more information on percolation see ref. 8.

The following definitions relating to entanglement are taken from ref. 3. For an edge $e = \{x, y\} \in \mathbb{L}$ we denote by $\langle e \rangle$ the closed line segment

$$\langle e \rangle = \{\lambda x + (1-\lambda)\, y\colon \lambda \in [0, 1]\} \subseteq \mathbb{R}^3.$$

For a graph G we write $[G] = \bigcup_{e \in G} \langle e \rangle \subseteq \mathbb{R}^3$. A *sphere* is a piecewise-linear subset of $\mathbb{R}^3$ which is homeomorphic to a topological 2-sphere. If S is a sphere, $\mathbb{R}^3 \setminus S$ has two path-components, one bounded and one unbounded. We call these the *inside* and the *outside* of S respectively. We say that a sphere S *separates* a set $R \subseteq \mathbb{R}^3$ if R intersects both the inside and the outside of S but not S itself. We say that a *finite* graph F is *entangled* if it is separated by no sphere, and we define

$$\mathscr{F} = \{F \subseteq \mathbb{L}\colon F \text{ is finite and entangled}\}.$$

If $\mathscr{A}$ is a set of graphs, by an $\mathscr{A}$*-graph* we mean a graph lying in $\mathscr{A}$, and by an $\mathscr{A}$*-subgraph* of a graph we mean a subgraph which is an $\mathscr{A}$-graph. An $\mathscr{A}$*-component* of a graph G is a maximal $\mathscr{A}$-subgraph of G. We define

$$\begin{aligned}\mathscr{E}^0 = \{G \subseteq \mathbb{L}\colon &\text{ every finite subgraph of } G \\ &\text{ is contained in some } \mathscr{F}\text{-subgraph of } G\}\end{aligned}$$

and

$$\mathscr{E}^1 = \{G \subseteq \mathbb{L}\colon G \text{ is separated by no sphere}\}.$$

As remarked in refs. 3 and 7, both $\mathscr{E}^0$ and $\mathscr{E}^1$ are natural candidates for a definition of the set of all "entangled" graphs, and they correspond in a natural way to "free" and "wired" boundary conditions respectively. We have that $\mathscr{E}^0 \subseteq \mathscr{E}^1$, but the two sets are not equal. See refs. 3 and 7 for more details.

For $i = 0, 1$ we define

$$\eta^i(p) = P_p(W \text{ has an infinite } \mathscr{E}^i\text{-component containing } O)$$

and

$$p_{\mathrm{e}}^{i} = \sup\{p \colon \eta^{i}(p) = 0\}.$$

It is immediate that

$$p_{\mathrm{e}}^{1} \leqslant p_{\mathrm{e}}^{0} \leqslant p_{\mathrm{c}}.$$

Theorem 1. We have the strict inequality

$$p_{\mathrm{e}}^{0} < p_{\mathrm{c}}.$$

Our proof of Theorem 1 is based on the argument in ref. 1, but there are some additional topological details.

Let E^0 be the $\mathscr{E}^0$-component of W containing O (that this is well-defined is shown in ref. 3). For a graph G we denote by $|G|$ the number of edges in G.

Theorem 2. Assume that $p_{\mathrm{e}}^{1} < p_{\mathrm{e}}^{0}$. Then for all $p \in (p_{\mathrm{e}}^{1}, p_{\mathrm{e}}^{0})$ there exists $\beta = \beta(p) < \infty$ such that

$$P_p(|E^0| \geqslant n) \geqslant \exp(-\beta n^{2/3})$$

for all $n \geqslant 1$.

Theorem 2 implies in particular that, under the hypothesis $p_{\mathrm{e}}^{1} < p_{\mathrm{e}}^{0}$, the distribution of $|E^0|$ cannot have exponentially decaying tails for all $p < p_{\mathrm{e}}^{0}$. This would be in contrast with the situation for connectivity percolation (see Theorem 6.75 in ref. 8). Our proof of Theorem 2 is based on the approach in the proof of Theorem 8.61 of ref. 8.

3. PROOFS

Proof of Theorem 1. We shall make use of an enhancement suggested in ref. 1 (see also ref. 8, p. 65). Let H be the subgraph of $\mathbb{L}$ consisting of all edges having both vertices in the box $[0, 2]^2 \times [0, 3]$. Let L be the subgraph of H consisting of the thick solid edges in Fig. 1 (the outline of the box is also illustrated). Let f be the dashed edge in Fig. 1. We define an enhancement of percolation configurations as follows. Given a configuration ω and the corresponding graph $W(\omega)$ we define

$$W' = W \cup \bigcup_{x \in E(W)} (f + x),$$

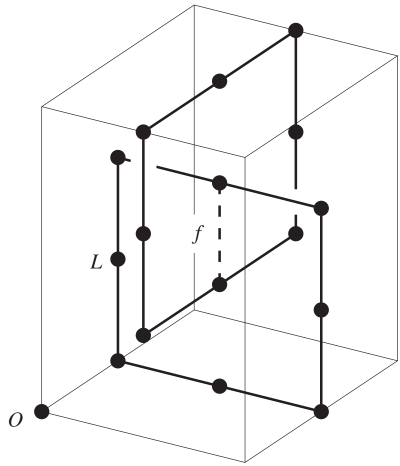

Fig. 1. The graph governing the enhancement.

where

$$E(W) = \{x \in \mathbb{Z}^3 \colon W \cap (H+x) = L+x\}.$$

Thus, W' is obtained from W in the following way. Wherever we see a translated copy of L in W, we add the edge corresponding to f. We refer to these added edges as *fasteners*. The idea is that the enhancement from W to W' affects the connectivity properties of the graph but has no effect on its entanglement properties.

It follows from the general results in that ref. 1 that there exists an interval $[p_1, p_2]$, where $p_1 < p_2$, such that for $p \in [p_1, p_2]$ we have

$$P_p(W \text{ has an infinite connected component}) = 0 \tag{1}$$

but

$$P_p(W' \text{ has an infinite connected component}) = 1. \tag{2}$$

Firstly, note that (1) implies that $p_2 \leqslant p_c$. Secondly, we claim that if W' has an infinite connected component, then W has an infinite $\mathscr{E}^0$-component. Using this fact, (2) implies that $p_e^0 \leqslant p_1$, and the required inequality then follows.

To prove the above claim, suppose that A' is an infinite connected component of W'. We shall adopt the following convention: graphs represented by primed letters are subgraphs of W', and graphs represented by unprimed letters are subgraphs of W. Let $A = A' \cap W$; A is A' with all fasteners removed. Clearly A is infinite. We shall show that $A \in \mathscr{E}^0$. Let B be a finite subgraph of A. We must show that B is a subgraph of some $\mathscr{F}$-subgraph of A. Since $B \subseteq A'$, let C' be a finite connected subgraph of A' which has B as a subgraph. We now let D' be the graph satisfying $C' \subseteq D' \subseteq A'$ obtained by enlarging C' so as to contain only complete copies of $L \cup \{f\}$; more formally, we define $D' = C' \cup \bigcup_x [(L \cup \{f\}) + x]$ where the union is over all $x \in E(W)$ satisfying $C' \cap ((L \cup \{f\}) + x) \neq \varnothing$. Note that D' is finite and connected. Let $D = D' \cap W$; D is D' with all fasteners removed; and note that $D \subseteq A$. We shall show that $D \in \mathscr{F}$, which will complete the proof.

Suppose that, contrary to the above statement, $[D]$ is separated by a sphere S. Each vertex in D lies in either the inside or the outside of S, and no edge of D can join a vertex in the inside to a vertex in the outside. Since D' is connected, it is easy to see that there exists some fastener in $D' \setminus D$ which joins a vertex in the inside of S to a vertex in the outside of S (consider adding the fasteners one at a time). Hence there exists some $x \in \mathbb{Z}^3$ such that $L + x \in D$ and such that S separates $[L + x]$. However, this is impossible: standard topological methods (using linking number, see ref. 9) may be used to prove that no sphere separates $[L]$. ∎

Proof of Theorem 2. We shall prove that the conclusion of the theorem holds for all p such that $\eta^1(p) > 0$. Let $\eta^1(p) > 0$. As noted in refs. 3 and 6, W has almost surely exactly one infinite $\mathscr{E}^1$-component, and we denote it I. Let $\zeta = \zeta(p)$ be the probability that a fixed edge e lies in I; by the FKG inequality $\zeta(p) \geqslant p\eta^1(p)$. For a positive integer m, let $B(m)$ be the graph consisting of all edges of $\mathbb{L}$ having both vertices in $[-m, m]^3$, and let $\partial B(m)$ be the graph of all edges having both vertices in $[-m, m]^3 \setminus [-m+1, m-1]^3$. We write $I_m = I \cap B(m)$. We claim that

$$P_p(|I_m| \geqslant \tfrac{1}{2}\zeta(p)|B(m)|) \geqslant \tfrac{1}{2}\zeta(p).$$

This is proved as follows. We have $|I_m| \leqslant |B(m)|$, and hence

$$\begin{aligned} E_p(|I_m|) &\leqslant P_p(|I_m| \geqslant \tfrac{1}{2}\zeta|B(m)|)\,|B(m)| + P_p(|I_m| < \tfrac{1}{2}\zeta|B(m)|)\,\tfrac{1}{2}\zeta|B(m)| \\ &\leqslant P_p(|I_m| \geqslant \tfrac{1}{2}\zeta|B(m)|)\,|B(m)| + \tfrac{1}{2}\zeta|B(m)|. \end{aligned}$$

The claim now follows from the fact that $E_p(|I_m|) = \zeta|B(m)|$.

Now define the event

$$Z_m = \{|I_m| \geqslant \tfrac{1}{2}\zeta(p)|B(m)|\} \cap \{I \text{ contains } O\} \cap \{\partial B(m) \text{ is open}\}.$$

Since each of the three events appearing in this definition is increasing, the FKG inequality gives

$$P_p(Z_m) \geqslant \tfrac{1}{2}\zeta(p)\,\eta^1(p)\,p^{|\partial B(m)|}.$$

Now, there exist positive constants a and b such that $|B(m)| \geqslant am^3$ and $|\partial B(m)| \leqslant bm^2$. We claim that whenever Z_m occurs, $I_m \cup \partial B(m)$ is an open $\mathscr{F}$-graph containing O. This claim implies the required result as follows. If Z_m occurs then we have $E^0 \supseteq I_m \cup \partial B(m)$, and so provided $\eta^1(p) > 0$ the above inequality yields

$$P_p(|E^0| \geqslant c(p)\, m^3) \geqslant \exp(-d(p)\, m^2)$$

where $c(p)$ and $d(p)$ are positive constants. The result follows on making a suitable choice of m.

The above claim follows from a special case of an argument used in the proof of Theorem 3.3 of ref. 3, with the topological boundary of the box $[-m, m]^3$ playing the role of the set "$[U]$" in the notation of ref. 3. The idea is to show that if $I_m \cup \partial B(m)$ were separated by a sphere S, we could modify it to obtain a sphere which separated $I_m \cup \partial B(m)$ but lay entirely in $[-m, m]^3$, giving a contradiction to the definition of I_m. This is done by repeatedly applying topological "surgery" to S so as to remove each intersection of S with the boundary of $[-m, m]^3$. For the details see ref. 3. Similar arguments appear in ref. 4, and detailed justification of the topological steps may be found in ref. 9. ∎

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